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Complicated invariant sets of some ordinary differential equations

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1 Introduction

The set of strange attractor for some two-dimensional system of ordinary differential equations seems to be connected but not arcwisely connected [Fig.3, p.124,5]. In this note we shall consider the nonlinear Mathieu equation :

$$\ddot{x} + k\dot{x} + a(t)x + x^3 = 0\,, \tag{1}$$

where k is a positive constant and $a(t)$ is continuous and π-periodic for t, which is equivalent to

$$\dot{x} = y\,, \quad \dot{y} = -ky - a(t)x - x^3\,, \tag{2}$$

[2], [p.22, 5]. Solutions of (1), and hence of (2) are uniformly ultimately bounded [3], and therefore Poincare's mapping of (2), say T, has an attracting set M, which is a maximal invariant, compact and connected set. Furthermore we shall assume that the trivial solution of (2) is directly unstable, namely one of the characteristic exponents is negative and another one is positive. Therefore, first of all, by the application of index of T, there exists a nontrivial π-periodic solutions of (2), which implies the existence of nontrivial fixed poins of T, and secondly, by [Theorem 4.1, 4.2, 1], T has a unstable, 1-dimensional analytic manifold around origin, which has the tangent. Cleary M contains any fixed point and any unstable manifold. Our result is the following.

Theorem 1
M is not arcwisely connected.

Remark. In the case where $a(t) = 1+\varepsilon \cos 2t$, $k = \varepsilon\lambda$ for constant λ, $0 < \lambda < \frac{1}{2}$, $\varepsilon > 0$ and ε is sufficiently small, the trivial solution is directly unstable.

2 Proof of Theorem 1

First of all, we shall consider the linear part of (2)

$$\dot{x} = y, \quad \dot{y} = -ky - a(t)x\,, \tag{3}$$

and represent this system by polar coordinate system : $x = r\cos\theta$, $y = r\sin\theta$ in the following

$$\frac{d\theta}{dt} = h(t,\theta) := -\sin^2\theta - k\sin\theta\cos\theta - a(t)\cos^2\theta\,, \tag{4}$$

We shall set $X(t)$ to be the fundamental matrix of (3), and hence since the trivial solution of (3) is directly unstable, $X(\pi)$ has two real independent eigenvectors. Therefore we can see that (4) has two second kind of periodic solutions $\theta_1(t)$ and $\theta_2(t)$ such that $\theta_1(t+\pi) = \theta_1(t)+n\pi$ and $\theta_2(t+\pi) = \theta_2(t) + n\pi$ for integer n. Similarly we shall represent (2) by polar coordinate system in the following

$$\frac{d\theta}{dt} = h(t,\theta) - 3r^2\cos\ 4\theta\,, \tag{5}$$

where $r^2(t) = x^2(t) + \dot{x}^2(t)$ and $x(t)$ is a nontrivial π-periodic solution. The existence of this periodic solution implies that (5) has a second kind of periodic solution $\theta(t)$ such that $\theta(t+\pi) = \theta(t) + m\pi$ for integer m. In the following we shall show that

$$n \neq m\,. \tag{6}$$

To the contrary, suppose that $n = m$, which implies that $\theta_1(t) - \theta(t)$ and $\theta_2(t) - \theta(t)$ are π-periodic for t. Therefore, we may see that $\cos(2\theta(t) + 2s(\theta_i(t) - \theta(t))$ and $\sin(2\theta(t) + 2s(\theta_i(t) - \theta(t))$ are π-periodic for t, where $i = 1,2$ and s is a number. Thus, setting

$$A(t) = \int_0^1 \frac{\partial h}{\partial\theta}(t, \theta(t) + s(\theta_i(t) - \theta(t))\,ds\,, \tag{7}$$

where $i = 1, 2$ and $\frac{\partial h}{\partial \theta}(t, \theta) = -k \cos 2\theta + (a(t) - 1) \sin 2\theta$, we may see that $A(t)$ is π-periodic for t. We can assume that either $\theta_1(0) \geq \theta(0)$ or $\theta_2(0) \geq \theta(0)$. Here we shall take the former case where $\theta_1(0) \geq \theta(0)$, because the latter case where $\theta_2(0) \geq \theta(0)$ can be exactly treated similarly. Setting $\xi(t) = \theta_1(t) - \theta(t)$, we can get

$$\dot{\xi} = A(t)\xi + r^2 \cos^4 \theta \tag{8}$$

where $A(t)$ is the function of (7) for $i = 1$, and hence

$$\xi(t) = \xi(0)e^{p(t)} + \int_0^t \left(e^{p(t)-p(s)}\right) r^2(s) \cos^4 \theta(s) ds \,, \tag{9}$$

where $p(t) = \int_0^t A(s)ds$. Since $\xi(0) \geq 0$, the first term of the right hand side is nonnegative and the second one is positive for $t > 0$. If $\xi(0) = 0$, then (9) is reduced to

$$\xi(t) = \int_0^t \left(e^{p(t)-p(s)}\right) r^2(s) \cos^4 \theta(s) ds \,, \tag{10}$$

which implies that $\xi(\pi) > 0$, and hence this is a contradiction to that $\xi(\pi) = \xi(0) = 0$. Thus we must have the case where $\xi(0) > 0$. In this case, if the mean value is nonzero, that is, if $p(\pi) \neq 0$, then $p(t) \to +\infty$ as either $t \to +\infty$ or $t \to -\infty$, and hence it follows from (9) that $\xi(t)$ is unbounded, which is a contradiction to the periodicity of $\xi(t)$. Therefore we must have that $p(\pi) = 0$, and hence by (9),

$$\xi(\pi) = \xi(0) + \int_0^{\pi} e^{-p(s)} r^2(s) \cos^4 \theta(s) ds \,. \tag{11}$$

Thus we obtain that $\xi(\pi) > \xi(0)$, which is also a contradiction to that $\xi(\pi) = \xi(0)$, and hence (6) is proved.

Now we shall show that M is not arcwisely connected. Let O denote the origin and P_0 the point of initial values of a nontrivial periodic solutions of (3). Since O is directly unstable, there exists an invariant unstable manifold around O, say l, such that l is an analytic curve through O and has a tangent. Taking an arbitrary point Q of l, we shall show that Q is not arcwisely connected to P_0 in Ω. To the contrary suppose that Q is arcwisely connected to P_0 in Ω. Then we can take a Jordan curve C' in Ω joining Q and P_0, and hence the Jordan curve C in Ω joining O and P_0, which is identical with l in a neighbourhood of O (see Figure 1).

Denoting by P the variable point from O to P_0 along C and by $\theta(P)$

the angle measured from the x-axis, which is continuous for $P \neq O$, we can see that $\theta(P)$ has a limit value as P approaches O. Therefore $\theta(P)$ is bounded on C, and hence $B(C) := \sup\{|\theta(P)|; P \in C\}$ is finite. Moreover, setting $C_j = T^j C$ for integer j, we can see that C_j is contained in M, and because of (6), that $|\theta_1(j\pi) - \theta(j\pi)| = |(n-m)j| \to \infty$ as $j \to \infty$. Thus we obtain that $B(C_j) > B(C)$ for large j, which implies that C_j is not identical with C for large j. Since both C_j and C joins O and P_0, we can take parts of C_j and C as a simple closed curve, say Γ_1, which is contained in M and whose inside domain is denoted by D_1. Furthermore we shall take a simple closed curve Γ_2, which contains M in its inside domain, say D_2. Since $T^j\Gamma_2$ never intersects M for $j \geq 1$, it follows that $T^j D_2 \supset D_1$ for j, which implies that $|T^j D_2| > |D_1|$ for $j \geq 1$, where $|D|$ denotes the area of domain D. On the other hand, because $k > 0$, it is known that $|T^j D_2| \to 0$ as $j \to \infty$. This contradiction completes the proof.

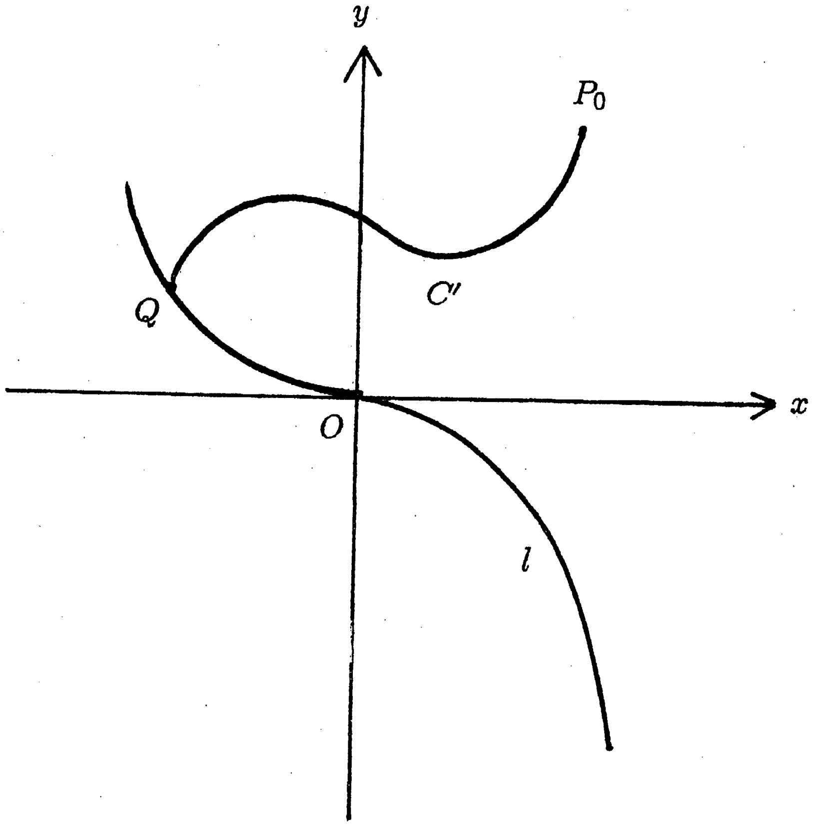

Figure 1

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